# TRANSFINITE DIAMETER NOTIONS IN $\mathbb{C}^N$ AND INTEGRALS OF VANDERMONDE DETERMINANTS

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Abstract. We provide a general framework and indicate relations between the notions of transfinite diameter, homogeneous transfinite diameter, and weighted transfinite diameter for sets in $\mathbb{C}^N$. An ingredient is a formula of Rumely [19] which relates the Robin function and the transfinite diameter of a compact set. We also prove limiting formulas for integrals of generalized Vandermonde determinants with varying weights for a general class of compact sets and measures in $\mathbb{C}^N$. Our results extend to certain weights and measures defined on cones in $\mathbb{R}^N$.

## 1. Introduction.

Given a compact set $E$ in the complex plane $\mathbb{C}$, the *transfinite diameter* of $E$ is the number

$$d(E) := \lim_{n\to\infty} \max_{\zeta_1,...,\zeta_n\in E} |VDM(\zeta_1,...,\zeta_n)|^{1/\binom{n}{2}} := \max_{\zeta_1,...,\zeta_n\in E} \prod_{i<j} |\zeta_i - \zeta_j|^{1/\binom{n}{2}}.$$

It is well-known that this quantity is equivalent to the *Chebyshev constant* of $E$:

$$T(E) := \lim_{n\to\infty} [\inf\{||p_n||_E : p_n(z) = z^n + \sum_{j=1}^{n-1} c_j z^j\}]^{1/n}$$

(here, $||p_n||_E := \sup_{z\in E} |p(z)|$) and also to $e^{-\rho(E)}$ where

$$\rho(E) := \lim_{|z|\to\infty} [g_E(z) - \log|z|]$$

is the *Robin constant* of $E$. The function $g_E$ is the *Green function* of logarithmic growth associated to $E$. Moreover, if $w$ is an admissible weight function on $E$, weighted versions of the above quantities can be defined. We refer the reader to the book of Saff-Totik [20] for the definitions and relationships.

For $E \subset \mathbb{C}^N$ with $N > 1$, multivariate notions of transfinite diameter, Chebyshev constant and Robin-type constants have been introduced and

*Date*: December 15, 2007.

1991 *Mathematics Subject Classification.* 32U20, 42C05.

*Key words and phrases.* Christoffel function, transfinite diameter.

*Supported in part by NSERC grant.

studied by several people. For an introduction to weighted versions of some of these quantitites, see Appendix B by Bloom in [20]. In the first part of this paper (section 2), we discuss a general framework for the various types of transfinite diameters in the spirit of Zaharjuta [23]. In particular, we relate (Theorem 2.7) two weighted transfinite diameters, $d^w(E)$ and $\delta^w(E)$, of a compact set $E \subset \mathbb{C}^N$ using a remarkable result of Rumely [19] which itself relates the (unweighted) transfinite diameter $d(E)$ with a Robin-like integral formula. Very recently Berman-Boucksom [2] have established a generalization of Rumely's formula which includes a weighted version of his result.

In the second part of the paper (section 3) we generalize to $\mathbb{C}^N$ some results on strong asymptotics of Christoffel functions proved in [8] in one variable. For $E$ a compact subset of $\mathbb{C}$, $w$ an admissible weight function on $E$, and $\mu$ a positive Borel measure on $E$ such that the triple $(E, w, \mu)$ satisfies a weighted Bernstein-Markov inequality (see (3.5)), we take, for each $n = 1, 2, ...,$ a set of orthonormal polynomials $q_1^{(n)}, ..., q_n^{(n)}$ with respect to the varying measures $w(z)^{2n} d\mu(z)$ where $\deg q_j^{(n)} = j - 1$ and form the sequence of Christoffel functions $K_n(z) := \sum_{j=1}^n |q_j^{(n)}(z)|^2$. In [8] we showed that

$$\frac{1}{n} K_n(z) w(z)^{2n} d\mu(z) \to d\mu_{eq}^w(z) \tag{1.1}$$

weak-* where $\mu_{eq}^w$ is the potential-theoretic weighted equilibrium measure. The key ingredients to proving (1.1) are, firstly, the verification that

$$\lim_{n \to \infty} Z_n^{1/n^2} = \delta^w(E) \tag{1.2}$$

where

$$Z_n = Z_n(E, w, \mu) := \tag{1.3}$$

$$\int_{E^n} |VDM(\lambda_1, ..., \lambda_n)|^2 w(\lambda_1)^{2n} \cdots w(\lambda_n)^{2n} d\mu(\lambda_1) \cdots d\mu(\lambda_n);$$

and, secondly, a "large deviation" result in the spirit of Johansson [16]. We generalize these two results to $\mathbb{C}^N$, $N > 1$ (Theorems 3.1 and 3.2). The methods are similar to the corresponding one variable methods and were announced in [8], Remark 3.1. In particular, $\delta^w(E)$ is interpreted as the transfinite diameter of a circled set in one higher dimension. We also discuss the case where $E = \Gamma$ is an unbounded cone in $\mathbb{R}^N$ for special weights and measures. Some of our results were proved independently by Berman-Boucksom [2].

We end the paper with a short section which includes questions related to these topics. We are grateful to Robert Berman for making reference [2]

available to us. The second author would also like to thank Sione Ma'u and Laura DeMarco for helpful conversations.

## 2. Transfinite diameter notions in $\mathbb{C}^N$.

We begin by considering a function $Y$ from the set of multiindices $\alpha \in \mathbf{N}^N$ to the nonnegative real numbers satisfying:

$$Y(\alpha+\beta) \leq Y(\alpha)\cdot Y(\beta) \text{ for all } \alpha,\ \beta \in \mathbf{N}^N. \tag{2.1}$$

We call a function $Y$ satisfying (2.1) *submultiplicative*; we have three main examples below. Let $e_1(z),...,e_j(z),...$ be a listing of the monomials $\{e_i(z) = z^{\alpha(i)} = z_1^{\alpha_1}\cdots z_N^{\alpha_N}\}$ in $\mathbb{C}^N$ indexed using a lexicographic ordering on the multiindices $\alpha = \alpha(i) = (\alpha_1,...,\alpha_N) \in \mathbf{N}^N$, but with $\deg e_i = |\alpha(i)|$ nondecreasing. We write $|\alpha| := \sum_{j=1}^N \alpha_j$.

We define the following integers:

(1) $m_d^{(N)} = m_d :=$ the number of monomials $e_i(z)$ of degree at most $d$ in $N$ variables;
(2) $h_d^{(N)} = h_d :=$ the number of monomials $e_i(z)$ of degree exactly $d$ in $N$ variables;
(3) $l_d^{(N)} = l_d :=$ the sum of the degrees of the $m_d$ monomials $e_i(z)$ of degree at most $d$ in $N$ variables.

We have the following relations:

$$m_d^{(N)} = \binom{N+d}{d};\ h_d^{(N)} = m_d^{(N)} - m_{d-1}^{(N)} = \binom{N-1+d}{d} \tag{2.2}$$

and

$$h_d^{(N+1)} = \binom{N+d}{d} = m_d^{(N)};\ l_d^{(N)} = N\binom{N+d}{N+1} = (\frac{N}{N+1})\cdot d m_d^{(N)}. \tag{2.3}$$

The elementary fact that the dimension of the space of homogeneous polynomials of degree $d$ in $N+1$ variables equals the dimension of the space of polynomials of degree at most $d$ in $N$ variables will be crucial in sections 4 and 5. Finally, we let

$$r_d^{(N)} = r_d := d h_d^{(N)} = d(m_d^{(N)} - m_{d-1}^{(N)})$$

which is the sum of the degrees of the $h_d$ monomials $e_i(z)$ of degree exactly $d$ in $N$ variables. We observe that

$$l_d^{(N)} = \sum_{k=1}^{d} r_k^{(N)} = \sum_{k=1}^{N} k h_k^{(N)}. \tag{2.4}$$

Let $K \subset \mathbb{C}^N$ be compact. Here are our three natural constructions associated to $K$:

(1) *Chebyshev constants*: Define the class of polynomials
$$P_i = P(\alpha(i)) := \{e_i(z) + \sum_{j<i} c_j e_j(z)\};$$
and the Chebyshev constants
$$Y_1(\alpha) := \inf\{||p||_K : p \in P_i\}.$$
We write $t_{\alpha,K} := t_{\alpha(i),K}$ for a Chebyshev polynomial; i.e., $t_{\alpha,K} \in P(\alpha(i))$ and $||t_{\alpha,K}||_K = Y_1(\alpha)$.

(2) *Homogeneous Chebyshev constants*: Define the class of homogeneous polynomials
$$P_i^{(H)} = P^{(H)}(\alpha(i)) := \{e_i(z) + \sum_{j<i,\ \deg(e_j)=\deg(e_i)} c_j e_j(z)\};$$
and the homogeneous Chebyshev constants
$$Y_2(\alpha) := \inf\{||p||_K : p \in P_i^{(H)}\}.$$
We write $t_{\alpha,K}^{(H)} := t_{\alpha(i),K}^{(H)}$ for a homogeneous Chebyshev polynomial; i.e., $t_{\alpha,K}^{(H)} \in P^{(H)}(\alpha(i))$ and $||t_{\alpha,K}^{(H)}||_K = Y_2(\alpha)$.

(3) *Weighted Chebyshev constants*: Let $w$ be an admissible weight function on $K$ (see below) and let
$$Y_3(\alpha) := \inf\{||w^{|\alpha(i)|}p||_K := \sup_{z\in K}\{|w(z)^{|\alpha(i)|}p(z)| : p \in P_i\}$$
be the weighted Chebyshev constants. Note we use the polynomial classes $P_i$ as in (1). We write $t_{\alpha,K}^w$ for a weighted Chebyshev polynomial; i.e., $t_{\alpha,K}^w$ is of the form $w^{\alpha(i)}p$ with $p \in P(\alpha(i))$ and $||t_{\alpha,K}^w||_K = Y_3(\alpha)$.

Let $\Sigma$ denote the standard $(N-1)-$simplex in $\mathbb{R}^N$; i.e.,
$$\Sigma = \{\theta = (\theta_1, ..., \theta_N) \in \mathbb{R}^N : \sum_{j=1}^N \theta_j = 1,\ \theta_j \geq 0,\ j = 1, ..., N\},$$
and let
$$\Sigma^0 := \{\theta \in \Sigma :\ \theta_j > 0,\ j = 1, ..., N\}.$$
Given a submultiplicative function $Y(\alpha)$, define, as with the above examples, a new function
$$\tau(\alpha) := Y(\alpha)^{1/|\alpha|}. \tag{2.5}$$

An examination of lemmas 1, 2, 3, 5, and 6 in [23] shows that (2.1) is the only property of the numbers $Y(\alpha)$ needed to establish those lemmas. That is, we have the following results for $Y : \mathbf{N}^N \to \mathbb{R}^+$ satisfying (2.1) and the associated function $\tau(\alpha)$ in (2.5).

**Lemma 2.1.** *For all $\theta \in \Sigma^0$, the limit*

$$T(Y,\theta) := \lim_{\alpha/|\alpha|\to\theta} Y(\alpha)^{1/|\alpha|} = \lim_{\alpha/|\alpha|\to\theta} \tau(\alpha)$$

*exists.*

**Lemma 2.2.** *The function $\theta \to T(Y,\theta)$ is log-convex on $\Sigma^0$ (and hence continuous).*

**Lemma 2.3.** *Given $b \in \partial\Sigma$,*

$$\liminf_{\theta\to b,\ \theta\in\Sigma^0} T(Y,\theta) = \liminf_{i\to\infty,\ \alpha(i)/|\alpha(i)|\to b} \tau(\alpha(i)).$$

**Lemma 2.4.** *Let $\theta(k) := \alpha(k)/|\alpha(k)|$ for $k = 1, 2, ...$ and let $Q$ be a compact subset of $\Sigma^0$. Then*

$$\limsup_{|\alpha|\to\infty}\{\log\tau(\alpha(k)) - \log T(Y(\theta(k))) : |\alpha(k)| = \alpha,\ \theta(k) \in Q\} = 0.$$

**Lemma 2.5.** *Define*

$$\tau(Y) := \exp\Big[\frac{1}{meas(\Sigma)}\int_\Sigma \log T(Y,\theta)d\theta\Big]$$

*Then*

$$\lim_{d\to\infty} \frac{1}{h_d}\sum_{|\alpha|=d} \log\tau(\alpha) = \log\tau(Y);$$

*i.e., using (2.5),*

$$\lim_{d\to\infty}\Big[\prod_{|\alpha|=d} Y(\alpha)\Big]^{1/dh_d} = \tau(Y).$$

One can incorporate all of the $Y(\alpha)'$s for $|\alpha| \leq d$; this is the content of the next result.

**Theorem 2.6.** *We have*

$$\lim_{d\to\infty}\Big[\prod_{|\alpha|\leq d} Y(\alpha)\Big]^{1/l_d} \text{ exists and equals } \tau(Y).$$

*Proof.* Define the geometric means

$$\tau_d^0 := \Big(\prod_{|\alpha|=d} \tau(\alpha)\Big)^{1/h_d},\ d = 1, 2, ...$$

The sequence

$$\log \tau_1^0, \log \tau_1^0, ...(r_1 \text{ times}), ..., \log \tau_d^0, \log \tau_d^0, ...(r_d \text{ times}), ...$$

converges to $\log \tau(Y)$ by the previous lemma; hence the arithmetic mean of the first $l_d = \sum_{k=1}^d r_k$ terms (see (2.4)) converges to $\log \tau(Y)$ as well. Exponentiating this arithmetic mean gives

$$(2.6) \qquad \big(\prod_{k=1}^d (\tau_k^0)^{r_k}\big)^{1/l_d} = \big(\prod_{k=1}^d \prod_{|\alpha|=k} \tau(\alpha)^k\big)^{1/l_d} = \big(\prod_{|\alpha|\leq d} Y(\alpha)\big)^{1/l_d}$$

and the result follows. □

Returning to our examples (1)-(3), example (1) was the original setting of Zaharjuta [23] which he utilized to prove the existence of the limit in the definition of the *transfinite diameter* of a compact set $K \subset \mathbb{C}^N$. For $\zeta_1, ..., \zeta_n \in \mathbb{C}^N$, let

$$(2.7) \qquad VDM(\zeta_1, ..., \zeta_n) = \det[e_i(\zeta_j)]_{i,j=1,...,n}$$

$$= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_n) \\ \vdots & \vdots & \ddots & \vdots \\ e_n(\zeta_1) & e_n(\zeta_2) & \ldots & e_n(\zeta_n) \end{bmatrix}$$

and for a compact subset $K \subset \mathbb{C}^N$ let

$$V_n = V_n(K) := \max_{\zeta_1,...,\zeta_n \in K} |VDM(\zeta_1, ..., \zeta_n)|.$$

Then

$$(2.8) \qquad d(K) = \lim_{d\to\infty} V_{m_d}^{1/l_d}$$

is the *transfinite diameter* of $K$; Zaharjuta [23] showed that the limit exists by showing that one has

$$(2.9) \qquad d(K) = \exp\big[\frac{1}{\text{meas}(\Sigma)} \int_{\Sigma^0} \log \tau(K, \theta) d\theta\big]$$

where $\tau(K, \theta) = T(Y_1, \theta)$ from (1); i.e., the right-hand-side of (2.9) is $\tau(Y_1)$. This follows from Theorem 2.6 for $Y = Y_1$ and the estimate

$$\big(\prod_{k=1}^d (\tau_k^0)^{r_k}\big)^{1/l_d} \leq V_{m_d}^{1/l_d} \leq (m_d!)^{1/l_d} \big(\prod_{k=1}^d (\tau_k^0)^{r_k}\big)^{1/l_d}$$

in [23] (compare (2.6)).

For a compact *circled* set $K \subset \mathbb{C}^N$; i.e., $z \in K$ if and only if $e^{i\phi} z \in K$, $\phi \in [0, 2\pi]$, one need only consider homogeneous polynomials in the definition of

the directional Chebyshev constants $\tau(K,\theta)$. In other words, in the notation of (1) and (2), $Y_1(\alpha)=Y_2(\alpha)$ for all $\alpha$ so that

$$T(Y_1,\theta)=T(Y_2,\theta) \text{ for circled sets } K.$$

This is because for such a set, if we write a polynomial $p$ of degree $d$ as $p=\sum_{j=0}^d H_j$ where $H_j$ is a homogeneous polynomial of degree $j$, then, from the Cauchy integral formula, $||H_j||_K \leq ||p||_K,\ j=0,...,d$. Moreover, a slight modification of Zaharjuta's arguments prove the existence of the limit of appropriate roots of maximal *homogeneous* Vandermonde determinants; i.e., the homogeneous transfinite diameter $d^{(H)}(K)$ of a compact set (cf., [15]). From the above remarks, it follows that

$$\text{for circled sets } K,\ d(K)=d^{(H)}(K). \tag{2.10}$$

Since we will be using the homogeneous transfinite diameter, we amplify the discussion. We relabel the standard basis monomials $\{e_i^{(H,d)}(z)=z^{\alpha(i)}=z_1^{\alpha_1}\cdots z_N^{\alpha_N}\}$ where $|\alpha(i)|=d,\ i=1,...,h_d$, we define the $d-$homogeneous Vandermonde determinant

$$VDMH_d((\zeta_1,...,\zeta_{h_d}):=\det\big[e_i^{(H,d)}(\zeta_j)\big]_{i,j=1,...,h_d}. \tag{2.11}$$

Then

$$d^{(H)}(K)=\lim_{d\to\infty}\big[\max_{\zeta_1,...,\zeta_{h_d}\in K}|VDMH_d(\zeta_1,...,\zeta_{h_d})|\big]^{1/dh_d} \tag{2.12}$$

is the homogeneous transfinite diameter of $K$; the limit exists and equals

$$\exp\big[\frac{1}{\text{meas}(\Sigma)}\int_{\Sigma^0}\log T(Y_2,\theta)d\theta\big]$$

where $T(Y_2,\theta)$ comes from (2).

Finally, related to example (3), there are similar properties for the weighted version of directional Chebyshev constants and transfinite diameter. To define weighted notions, let $K\subset\mathbf{C}^N$ be closed and let $w$ be an *admissible* weight function on $K$; i.e., $w$ is a nonnegative, usc function with $\{z\in K: w(z)>0\}$. Let $Q:=-\log w$ and define the weighted pluricomplex Green function $V_{K,Q}^*(z):=\limsup_{\zeta\to z}V_{K,Q}(\zeta)$ where

$$V_{K,Q}(z):=\sup\{u(z): u\in L(\mathbb{C}^N),\ u\leq Q \text{ on } K\}.$$

Here, $L(\mathbb{C}^N)$ is the set of all plurisubharmonic functions $u$ on $\mathbb{C}^N$ with the property that $u(z)-\log|z|=0(1),\ |z|\to\infty$. If $K$ is closed but not necessarily bounded, we require that $w$ satisfies the growth property

$$|z|w(z)\to 0 \text{ as } |z|\to\infty,\ z\in K, \tag{2.13}$$

so that $V_{K,Q}$ is well-defined and equals $V_{K\cap B_R,Q}$ for $R>0$ sufficiently large where $B_R=\{z:|z|\leq R\}$ (Definition 2.1 and Lemma 2.2 of Appendix B in

[20]). The unweighted case is when $w \equiv 1$ ($Q \equiv 0$); we then write $V_K$ for the pluricomplex Green function. The set $K$ is called *regular* if $V_K = V_K^*$; i.e., $V_K$ is continuous; and $K$ is *locally regular* if for each $z \in K$, the sets $K \cap \overline{B(z,r)}$ are regular for $r > 0$ where $B(z,r)$ denotes the ball of radius $r$ centered at $z$. We define the *weighted transfinite diameter*

$$d^w(K) := \exp\Big[\frac{1}{\text{meas}(\Sigma)} \int_{\Sigma^0} \log \tau^w(K,\theta) d\theta\Big]$$

as in [9] where $\tau^w(K,\theta) = T(Y_3,\theta)$ from (3); i.e., the right-hand-side of this equation is the quantity $\tau(Y_3)$.

We remark for future use that if $\{K_j\}$ is a decreasing sequence of locally regular compacta with $K_j \downarrow K$, and if $w_j$ is a continuous admissible weight function on $K_j$ with $w_j \downarrow w$ on $K$ where $w$ is an admissible weight function on $K$, then the argument in Proposition 7.5 of [9] shows that $\lim_{j\to\infty} \tau^{w_j}(K_j,\theta) = \tau^w(K,\theta)$ for all $\theta \in \Sigma^0$ (we mention that there is a misprint in the statement of this proposition in [9]) and hence

$$\lim_{j\to\infty} d^{w_j}(K_j) = d^w(K). \tag{2.14}$$

In particular, (2.14) holds in the unweighted case ($w \equiv 1$) for any decreasing sequence $\{K_j\}$ of compacta with $K_j \downarrow K$; i.e.,

$$\lim_{j\to\infty} d(K_j) = d(K) \tag{2.15}$$

(cf., [9] equation (1.13)).

Another natural definition of a weighted transfinite diameter uses weighted Vandermonde determinants. Let $K \subset \mathbb{C}^N$ be compact and let $w$ be an admissible weight function on $K$. Given $\zeta_1, ..., \zeta_n \in K$, let

$$\begin{aligned} W(\zeta_1,...,\zeta_n) &:= VDM(\zeta_1,...,\zeta_n) w(\zeta_1)^{|\alpha(n)|} \cdots w(\zeta_n)^{|\alpha(n)|} \\ &= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_n) \\ \vdots & \vdots & \ddots & \vdots \\ e_n(\zeta_1) & e_n(\zeta_2) & \ldots & e_n(\zeta_n) \end{bmatrix} \cdot w(\zeta_1)^{|\alpha(n)|} \cdots w(\zeta_n)^{|\alpha(n)|} \end{aligned}$$

be a *weighted Vandermonde determinant*. Let

$$W_n := \max_{\zeta_1,...,\zeta_n \in K} |W(\zeta_1,...,\zeta_n)| \tag{2.16}$$

and define an *$n-th$ weighted Fekete set for $K$ and $w$* to be a set of $n$ points $\zeta_1, ..., \zeta_n \in K$ with the property that

$$|W(\zeta_1,...,\zeta_n)| = \sup_{\xi_1,...,\xi_n \in K} |W(\xi_1,...,\xi_n)|.$$

Also, define

$$\delta^w(K) := \limsup_{d\to\infty} W_{m_d}^{1/l_d}. \tag{2.17}$$

We will show in Proposition 2.1 that $\lim_{d\to\infty} W_{m_d}^{1/l_d}$ (the weighted analogue of (2.8)) exists. The question of the existence of this limit if $N > 1$ was raised in [9]. Moreover, using a recent result of Rumely, we show how $\delta^w(K)$ is related to $d^w(K)$:

$$\delta^w(K) = [\exp(-\int_K Q(dd^cV^*_{K,Q})^N)]^{1/N} \cdot d^w(K) \tag{2.18}$$

where $(dd^cV^*_{K,Q})^N$ is the complex Monge-Ampere operator applied to $V^*_{K,Q}$. We refer the reader to [17] or Appendix B of [20] for more on the complex Monge-Ampere operator.

We begin by proving the existence of the limit in the definition of $\delta^w(E)$ in (2.1) for a set $E \subset \mathbb{C}^N$ and an admissible weight $w$ on $E$ (see also [2]).

**Proposition 2.1.** *Let $E \subset \mathbb{C}^N$ be a compact set with an admissible weight function $w$. The limit*

$$\delta^w(E) := \lim_{d\to\infty}\big[\max_{\lambda^{(i)}\in E} |VDM(\lambda^{(1)},...,\lambda^{(m_d^{(N)})})|\cdot w(\lambda^{(1)})^d \cdots w(\lambda^{(m_d^{(N)})})^d\big]^{1/l_d^{(N)}}$$

*exists.*

*Proof.* Following [6], we define the circled set

$$F = F(E,w) := \{(t,z) = (t,t\lambda) \in \mathbb{C}^{N+1} : \lambda \in E,\ |t| = w(\lambda)\}.$$

We first relate weighted Vandermonde determinants for $E$ with homogeneous Vandermonde determinants for $F$. To this end, for each positive integer $d$, choose

$$m_d^{(N)} = \binom{N+d}{d}$$

(recall (2.2)) points $\{(t_i, z^{(i)})\}_{i=1,...,m_d^{(N)}} = \{(t_i, t_i\lambda^{(i)})\}_{i=1,...,m_d^{(N)}}$ in $F$ and form the $d-$homogeneous Vandermonde determinant

$$VDMH_d((t_1,z^{(1)}),...,(t_{m_d^{(N)}}, z^{(m_d^{(N)})})).$$

We extend the lexicographical order of the monomials in $\mathbb{C}^N$ to $\mathbb{C}^{N+1}$ by letting $t$ precede any of $z_1,...,z_N$. Writing the standard basis monomials of degree $d$ in $\mathbb{C}^{N+1}$ as

$$\{t^{d-j}e_k^{(H,d)}(z) : j = 0,...,d;\ k = 1,...,h_j\};$$

i.e., for each power $d-j$ of $t$, we multiply by the standard basis monomials of degree $j$ in $\mathbb{C}^N$, and dropping the superscript $(N)$ in $m_d^{(N)}$, we have the

$d-$homogeneous Vandermonde matrix

$$\begin{bmatrix} t_1^d & t_2^d & \dots & t_{m_d}^d \\ t_1^{d-1}e_2(z^{(1)}) & t_2^{d-1}e_2(z^{(2)}) & \dots & t_{m_d}^{d-1}e_2(z^{(m_d)}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{m_d}(z^{(1)}) & e_{m_d}(z^{(2)}) & \dots & e_{m_d}(z^{(m_d)}) \end{bmatrix}$$

$$= \begin{bmatrix} t_1^d & t_2^d & \dots & t_{m_d}^d \\ t_1^{d-1}z_1^{(1)} & t_2^{d-1}z_1^{(2)} & \dots & t_{m_d}^{d-1}z_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (z_N^{(1)})^d & (z_N^{(2)})^d & \dots & (z_N^{(m_d)})^d \end{bmatrix}.$$

Factoring $t_i^d$ out of the $i-$th column, we obtain

$$VDMH_d((t_1,z^{(1)}),...,(t_{m_d},z^{(m_d)})) = t_1^d \cdots t_{m_d}^d \cdot VDM(\lambda^{(1)},...,\lambda^{(m_d)});$$

thus, writing $|A| := |\det A|$ for a square matrix $A$,

$$\begin{vmatrix} t_1^d & t_2^d & \dots & t_{m_d}^d \\ t_1^{d-1}z_1^{(1)} & t_2^{d-1}z_1^{(2)} & \dots & t_{m_d}^{d-1}z_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (z_N^{(1)})^d & (z_N^{(2)})^d & \dots & (z_N^{(m_d)})^d \end{vmatrix} \tag{2.19}$$

$$= |t_1|^d \cdots |t_{m_d}|^d \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m_d)} \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_N^{(1)})^d & (\lambda_N^{(2)})^d & \dots & (\lambda_N^{(m_d)})^d \end{vmatrix},$$

where $\lambda_k^{(j)} = z_k^{(j)}/t_j$ provided $t_j \neq 0$. By definition of $F$, since $(t_i, z^{(i)}) = (t_i, t_i\lambda^{(i)}) \in F$, we have $|t_i| = w(\lambda^{(i)})$ so that from (2.19)

$$VDMH_d((t_1,z^{(1)}),...,(t_{m_d},z^{(m_d)}))$$
$$= VDM(\lambda^{(1)},...,\lambda^{(m_d)}) \cdot w(\lambda^{(1)})^d \cdots w(\lambda^{(m_d)})^d.$$

Thus

$$\max_{(t_i,z^{(i)})\in F} |VDMH_d((t_1,z^{(1)}),...,(t_{m_d},z^{(m_d)}))| =$$
$$\max_{\lambda^{(i)}\in E} |VDM(\lambda^{(1)},...,\lambda^{(m_d)})| \cdot w(\lambda^{(1)})^d \cdots w(\lambda^{(m_d)})^d.$$

Note that the maximum will occur when all $t_j = w(\lambda^{(j)}) > 0$. As mentioned in section 3 the limit

$$\lim_{d\to\infty} \big[ \max_{(t_i,z^{(i)})\in F} |VDMH_d((t_1,z^{(1)}),...,(t_{m_d},z^{(m_d)}))| \big]^{1/dh_d^{(N+1)}} =: d^{(H)}(F)$$

exists [15]; thus the limit

$$\lim_{d\to\infty}\big[\max_{\lambda^{(i)}\in E}|VDM(\lambda^{(1)},...,\lambda^{(m_d)})|\cdot w(\lambda^{(1)})^d\cdots w(\lambda^{(m_d)})^d\big]^{1/l_d^{(N)}} := \delta^w(E)$$

exists. □

**Corollary 2.1.** *For $E\subset\mathbb{C}^N$ a nonpluripolar compact set with an admissible weight function $w$ and*

$$F = F(E,w) := \{(t,z)=(t,t\lambda)\in\mathbb{C}^{N+1} : \lambda\in E,\ |t| = w(\lambda)\},$$

$$\delta^w(E) = d^{(H)}(F)^{\frac{N+1}{N}} = d(F)^{\frac{N+1}{N}}. \tag{2.20}$$

*Proof.* The first equality follows from the proof of Proposition 2.1 using the relation

$$l_d^{(N)} = (\frac{N}{N+1})\cdot dh_d^{(N+1)}$$

(see (2.3)). The second equality is (2.10). □

We next relate $\delta^w(E)$ and $d^w(E)$ but we first recall the remarkable formula of Rumely [19]. For a plurisubharmonic function $u$ in $L(\mathbb{C}^N)$ we can define the *Robin function* associated to $u$:

$$\rho_u(z) := \limsup_{|\lambda|\to\infty}\left[u(\lambda z) - \log(|\lambda|)\right].$$

This function is plurisubharmonic (cf., [5], Proposition 2.1) and logarithmically homogeneous:

$$\rho_u(tz) = \rho_u(z) + \log|t| \text{ for } t\in\mathbb{C}.$$

For $u = V_{E,Q}^*$ $(V_E^*)$ we write $\rho_u = \rho_{E,Q}$ $(\rho_E)$. Rumely's formula relates $\rho_E$ and $d(E)$:

$$\begin{aligned} -\log d(E) = \frac{1}{N}\Big[&\int_{\mathbb{C}^{N-1}}\rho_E(1,t_2,...,t_N)(dd^c\rho_E(1,t_2,...,t_N))^{N-1} \\ &+\int_{\mathbb{C}^{N-2}}\rho_E(0,1,t_3,...,t_N)(dd^c\rho_E(0,1,t_3,...,t_N))^{N-2} \\ &+\cdots+\int_{\mathbb{C}}\rho_E(0,..,0,1,t_N)(dd^c\rho_E(0,..,0,1,t_N)+\rho_E(0,..,0,1)\Big]. \end{aligned} \tag{2.21}$$

Here we make the convention that $dd^c = \frac{1}{2\pi}(2i\partial\overline{\partial})$ so that in any dimension $d=1,2,...,$

$$\int_{\mathbb{C}^d}(dd^cu)^d = 1$$

for any $u\in L^+(\mathbb{C}^d)$; i.e., for any plurisubharmonic function $u$ in $\mathbb{C}^d$ which satisfies

$$C_1 + \log(1+|z|) \le u(z) \le C_2 + \log(1+|z|)$$

for some $C_1, C_2$.

We begin by rewriting (2.21) for *regular circled* sets $E$ using an observation of Sione Ma'u. Note that for such sets, $V_E^* = \rho_E^+ := \max(\rho_E, 0)$. If we intersect $E$ with a hyperplane $\mathcal{H}$ through the origin, e.g., by rotating coordinates, we take $\mathcal{H} = \{z = (z_1, ..., z_N) \in \mathbb{C}^N : z_1 = 0\}$, then $E \cap \mathcal{H}$ is a regular, compact, circled set in $\mathbb{C}^{N-1}$ (which we identify with $\mathcal{H}$). Moreover, we have

$$\rho_{\mathcal{H}\cap E}(z_2, ..., z_N) = \rho_E(0, z_2, ..., z_N)$$

since each side is logarithmically homogeneous and vanishes for $(z_2, ..., z_N) \in \partial(\mathcal{H} \cap E)$. Thus the terms

$$\int_{\mathbb{C}^{N-2}} \rho_E(0, 1, t_3, ..., t_N)(dd^c \rho_E(0, 1, t_3, ..., t_N))^{N-2}$$
$$+ \cdots + \int_{\mathbb{C}} \rho_E(0, .., 0, 1, t_N)(dd^c \rho_E(0, .., 0, 1, t_N) + \rho_E(0, .., 0, 1)$$

in (2.21) are seen to equal

$$(N-1)d^{\mathbb{C}^{N-1}}(\mathcal{H} \cap E)$$

(where we temporarily write $d^{\mathbb{C}^{N-1}}$ to denote the transfinite diameter in $\mathbb{C}^{N-1}$ for emphasis) by applying (2.21) in $\mathbb{C}^{N-1}$ to the set $\mathcal{H} \cap E$. Hence we have

$$-\log d(E) = \frac{1}{N}\int_{\mathbb{C}^{N-1}} \rho_E(1, t_2, ..., t_N)(dd^c \rho_E(1, t_2, ..., t_N))^{N-1} + \big(\frac{N-1}{N}\big)[-\log d^{\mathbb{C}^{N-1}}(\mathcal{H} \cap E)]. \tag{2.22}$$

**Theorem 2.7.** *For $E \subset \mathbb{C}^N$ a nonpluripolar compact set with an admissible weight function $w$,*

$$\delta^w(E) = [\exp(-\int_E Q(dd^c V_{E,Q}^*)^N)]^{1/N} \cdot d^w(E). \tag{2.23}$$

*Proof.* We first assume that $E$ is locally regular and $Q$ is continuous. It is known in this case that $V_{E,Q} = V_{E,Q}^*$ (cf., [21], Proposition 2.16). As before, we define the circled set

$$F = F(E, w) := \{(t, z) = (t, t\lambda) \in \mathbb{C}^{N+1} : \lambda \in E,\ |t| = w(\lambda)\}.$$

We claim this is a regular set; i.e., $V_F$ is continuous. First of all, $V_F^*(t, z) = \max[\rho_F(t, z), 0]$ (cf., Proposition 2.2 of [6]) so that it suffices to verify that $\rho_F(t, z)$ is continuous. From Theorem 2.1 and Corollary 2.1 of [6],

$$V_{E,Q}(\lambda) = \rho_F(1, \lambda) \text{ on } \mathbb{C}^N \tag{2.24}$$

which implies, by the logarithmic homogeneity of $\rho_F$, that $\rho_F(t,z)$ is continuous on $\mathbb{C}^{N+1}\setminus\{t=0\}$. Corollary 2.1 and equation (2.8) in [6] give that

$$\rho_F(0,\lambda)=\rho_{E,Q}(\lambda) \text{ for } \lambda\in\mathbb{C}^N \tag{2.25}$$

and $\rho_{E,Q}$ is continuous by Theorem 2.5 of [9]. Moreover, the limit exists in the definition of $\rho_{E,Q}$:

$$\rho_{E,Q}(\lambda):=\limsup_{|t|\to\infty}[V_{E,Q}(t\lambda)-\log|t|]=\lim_{|t|\to\infty}[V_{E,Q}(t\lambda)-\log|t|];$$

and the limit is uniform in $\lambda$ (cf., Corollary 4.4 of [11]) which implies, from (2.24) and (2.25), that $\lim_{t\to 0}\rho_F(t,\lambda)=\rho_F(0,\lambda)$ so that $\rho_F(t,z)$ is continuous. In particular,

$$V_{E,Q}(\lambda)=Q(\lambda)=\rho_F(1,\lambda) \text{ on the support of } (dd^cV_{E,Q})^N$$

so that

$$\int_E Q(\lambda)(dd^cV_{E,Q}(\lambda))^N=\int_{\mathbb{C}^N}\rho_F(1,\lambda)(dd^c\rho_F(1,\lambda))^N. \tag{2.26}$$

On the other hand, $E_\rho^w:=\{\lambda\in\mathbb{C}^N:\rho_{E,Q}(\lambda)\le 0\}$ is a circled set, and, according to eqn. (3.14) in [9], $d^w(E)=d(E_\rho^w)$. But

$$\rho_{E,Q}(\lambda)=\limsup_{|t|\to\infty}[V_{E,Q}(t\lambda)-\log|t|]$$

$$=\limsup_{|t|\to\infty}[\rho_F(1,t\lambda)-\log|t|]=\limsup_{|t|\to\infty}\rho_F(1/t,\lambda)=\rho_F(0,\lambda).$$

Thus

$$E_\rho^w=\{\lambda\in\mathbb{C}^N:\rho_F(0,\lambda)\le 0\}=F\cap\mathcal{H}$$

where $\mathcal{H}=\{(t,z)\in\mathbb{C}^{N+1}:t=0\}$ and hence

$$d^w(E)=d(E_\rho^w)=d(F\cap\mathcal{H}). \tag{2.27}$$

From (2.22) applied to $F\subset\mathbb{C}^{N+1}$,

$$-\log d(F)=\frac{1}{N+1}\int_{\mathbb{C}^N}\rho_F(1,\lambda)(dd^c\rho_F(1,\lambda))^N \tag{2.28}$$

$$+(\frac{N}{N+1})[-\log d(F\cap\mathcal{H})].$$

Finally, from (2.20),

$$\delta^w(E)=d(F)^{\frac{N+1}{N}}; \tag{2.29}$$

putting together (2.26), (2.27), (2.28) and (2.29) gives the result if $E$ is locally regular and $Q$ is continuous.

The general case follows from approximation. Take a sequence of locally regular compacta $\{E_j\}$ decreasing to $E$ and a sequence of weight functions

$\{w_j\}$ with $w_j$ continuous and admissible on $E_j$ and $w_j \downarrow w$ on $E$ (cf., Lemma 2.3 of [6]). From (2.14) we have

$$\lim_{j\to\infty} d^{w_j}(E_j) = d^w(E). \tag{2.30}$$

Also, by Corollary 2.1 we have

$$\delta^{w_j}(E_j) = d(F_j)^{\frac{N+1}{N}} \tag{2.31}$$

where

$$F_j = F_j(E_j, w_j) = \{(t(1,\lambda) : \lambda \in E_j,\ |t| = w_j(\lambda)\}.$$

Since $E_{j+1} \subset E_j$ and $w_{j+1} \leq w_j$, the sets

$$\tilde{F}_j = \tilde{F}_j(E_j, w_j) = \{(t(1,\lambda) : \lambda \in E_j,\ |t| \leq w_j(\lambda)\}$$

satisfy $\tilde{F}_{j+1} \subset \tilde{F}_j$ and hence

$$d(\tilde{F}_{j+1}) = d(F_{j+1}) \leq d(\tilde{F}_j) = d(F_j).$$

Since $F_j \downarrow F$, we conclude from (2.15) and (2.31) that

$$\lim_{j\to\infty} \delta^{w_j}(E_j) = \delta^w(E). \tag{2.32}$$

Applying (2.23) to $E_j, w_j, Q_j$ and using (2.30) and (2.32), we conclude that

$$\int_{E_j} Q_j (dd^c V_{E_j,Q_j})^N \to \int_E Q(dd^c V^*_{E,Q})^N,$$

completing the proof of (2.23). □

## 3. **Integrals of Vandermonde determinants.**

In this section, we first state and prove the analogue of an "unweighted" generalization to $\mathbb{C}^N$ of Theorem 2.1 of [8] as it has a self-contained proof. We first recall some terminology. Given a compact set $E \subset \mathbb{C}^N$ and a measure $\nu$ on $E$, we say that $(E,\nu)$ satisfies the Bernstein-Markov inequality for holomorphic polynomials in $\mathbb{C}^N$ if, given $\epsilon > 0$, there exists a constant $M = M(\epsilon)$ such that for all such polynomials $Q_n$ of degree at most $n$

$$||Q_n||_E \leq M(1+\epsilon)^n ||Q_n||_{L^2(\nu)}. \tag{3.1}$$

**Theorem 3.1.** *Let $(E,\mu)$ satisfy a Bernstein-Markov inequality. Then*

$$\lim_{d\to\infty} Z_d^{1/2l_d^{(N)}} = d(E)$$

*where*

$$Z_d = Z_d(E,\mu) := \tag{3.2}$$

$$\int_{E^{m_d^{(N)}}} |VDM(\lambda^{(1)},...,\lambda^{m_d^{(N)}})|^2 d\mu(\lambda^{(1)})\cdots d\mu(\lambda^{m_d^{(N)}}).$$

*Proof.* Since $VDM(\zeta_1,...,\zeta_n) = \det[e_i(\zeta_j)]_{i,j=1,...,n}$ for any positive integer $n$, if we apply the Gram-Schmidt procedure to the monomials $e_1,...,e_{m_d^{(N)}}$ to obtain orthogonal polynomials $q_1,...,q_{m_d^{(N)}}$ with respect to $\mu$ where $q_j \in P_j$ has minimal $L^2(\mu)-$norm among all such polynomials, we get, upon using elementary row operations on $VDM(\zeta_1,...,\zeta_{m_d^{(N)}})$ and expanding the determinant (cf., [14] Chapter 5 or section 2 of [8])

(3.3)
$$\int_{E^{m_d^{(N)}}} |VDM(\zeta_1,...,\zeta_{m_d^{(N)}})|^2 d\mu(\zeta_1)\cdots d\mu(\zeta_{m_d^{(N)}}) = m_d^{(N)}! \prod_{j=1}^{m_d^{(N)}} ||q_j||^2_{L^2(\mu)}.$$

Let $t_{\alpha,E} \in P(\alpha)$ be a Chebyshev polynomial; i.e., $||t_{\alpha,E}||_E = Y_1(\alpha)$. Then Theorem 2.6 shows that

$$\lim_{d\to\infty}\big(\prod_{|\alpha|\leq d} ||t_{\alpha,E}||_E\big)^{1/l_d} = \tau(Y_1)$$

since

$$\lim_{d\to\infty}(m_d^{(N)}!)^{1/l_d^{(N)}} = 1.$$

Zaharjuta's theorem (2.9) shows that $\tau(Y_1) = d(E)$ so we need show that

$$\lim_{d\to\infty}\big(\prod_{|\alpha|\leq d} ||t_{\alpha,E}||_E\big)^{1/l_d} = \lim_{d\to\infty}\big(\prod_{|\alpha|\leq d} ||q_\alpha||_{L^2(\mu)}\big)^{1/l_d}. \tag{3.4}$$

This follows from the Bernstein-Markov property. First note that

$$||q_\alpha||_{L^2(\mu)} \leq ||t_{\alpha,E}||_{L^2(\mu)} \leq \mu(E)\cdot ||t_{\alpha,E}||_E$$

from the $L^2(\mu)-$norm minimality of $q_\alpha$; then, given $\epsilon > 0$, the Bernstein-Markov property and the sup-norm minimality of $t_{\alpha,E}$ give

$$||t_{\alpha,E}||_E \leq ||q_\alpha||_E \leq M(1+\epsilon)^{|\alpha|}||q_\alpha||_{L^2(\mu)}$$

for some $M = M(\epsilon) > 0$. Taking products of these inequalities over $|\alpha| \leq d$; taking $l_d-$th roots; and letting $\epsilon \to 0$ gives the result. This reasoning is adapted from the proof of Theorem 3.3 in [7]. □

A weighted polynomial on $E$ is a function of the form $w(z)^n p_n(z)$ where $p_n$ is a holomorphic polynomial of degree at most $n$. Let $\mu$ be a measure with support in $E$ such that $(E, w, \mu)$ satisfies a Bernstein-Markov inequality for weighted polynomials (referred to as a *weighted B-M inequality* in [6]):

given $\epsilon > 0$, there exists a constant $M = M(\epsilon)$ such that for all weighted polynomials $w^n p_n$

$$||w^n p_n||_E \leq M(1+\epsilon)^n ||w^n p_n||_{L^2(\mu)}. \tag{3.5}$$

Generalizing Theorem 3.1, we have the following result.

**Theorem 3.2.** *Let $(E, w, \mu)$ satisfy a Bernstein-Markov inequality (3.5) for weighted polynomials. Then*

$$\lim_{d\to\infty} Z_d^{1/2l_d^{(N)}} = \delta^w(E)$$

*where*

$$Z_d = Z_d(E, w, \mu) := \int_{E^{m_d^{(N)}}} |VDM(\lambda^{(1)}, ..., \lambda^{(m_d^{(N)})})|^2 \times \tag{3.6}$$

$$w(\lambda^{(1)})^{2d} \cdots w(\lambda^{(m_d^{(N)})})^{2d} d\mu(\lambda^{(1)}) \cdots d\mu(\lambda^{(m_d^{(N)})}).$$

The proof of Theorem 3.2 follows along the lines of section 3 of [8]. Let $E \subset \mathbb{C}^N$ be a nonpluripolar compact set with an admissible weight function $w$ and let $\mu$ be a measure with support in $E$ such that $(E, w, \mu)$ satisfies a Bernstein-Markov inequality for weighted polynomials. The integrand

$$|VDM(\lambda^{(1)}, ..., \lambda^{(m_d^{(N)})})|^2 \cdot w(\lambda^{(1)})^{2d} \cdots w(\lambda^{(m_d^{(N)})})^{2d}$$

in the definition of $Z_d$ in (3.6) has a maximal value on $E^{m_d^{(N)}}$ whose $1/2l_d^{(N)}$ root tends to $\delta^w(E)$. To show that the integrals themselves have the same property, we begin by constructing the circled set $F \subset \mathbb{C}^{N+1}$ defined as in section 4:

$$F = F(E, w) := \{(t, z) = (t, t\lambda) \in \mathbb{C}^{N+1} : \lambda \in E,\ |t| = w(\lambda)\}.$$

We construct a measure $\nu$ on $F$ associated to $\mu$ such that $(F, \nu)$ satisfies the Bernstein-Markov property for holomorphic polynomials in $\mathbb{C}^{N+1}$; i.e., (3.1) holds. Define

$$\nu := m_\lambda \otimes \mu,\ \lambda \in E$$

where $m_\lambda$ is normalized Lebesgue measure on the circle $|t| = w(\lambda)$ in the complex $t-$plane given by

$$C_\lambda := \{(t, t\lambda) \in \mathbb{C}^{N+1} : t \in \mathbb{C}\}.$$

That is, if $\phi$ is continuous on $F$,

$$\int_F \phi(t, z) d\nu(t, z) = \int_E \big[\int_{C_\lambda} \phi(t, t\lambda) dm_\lambda(t)\big] d\mu(\lambda).$$

Equivalently, if $\pi : \mathbb{C}^{N+1} \to \mathbb{C}^N$ via $\pi(t, z) = z/t := \lambda$, then $\pi_*(\nu) = \mu$. The fact that $(F, \nu)$ satisfies the Bernstein-Markov property follows from

Theorem 3.1 of [6]. Moreover, if $p_1(t,z)$ and $p_2(t,z)$ are two homogeneous polynomials in $\mathbb{C}^{N+1}$ of degree $d$, say, and we write

$$p_j(t,z) = p_j(t,t\lambda) = t^d p_j(1,\lambda) =: t^d G_j(\lambda), \ j = 1,2$$

for univariate $G_j$, then it is straightforward to see that

$$\int_F p_1(t,z)\overline{p_2(t,z)}d\nu(t,z) = \int_E G_1(\lambda)\overline{G_2(\lambda)}w(\lambda)^{2d}d\mu(\lambda) \tag{3.7}$$

(cf., [6], Lemma 3.1 and its proof). Note that if

$$p(t,z) = t^i z^\alpha = t^i z_1^{\alpha_1}\cdots z_N^{\alpha_N}$$

with $|\alpha| = \alpha_1 + \cdots + \alpha_N = d - i$, then

$$p(t,z) = t^d(z/t)^\alpha = t^d G(\lambda) = t^d \cdot \lambda_1^{\alpha_1}\cdots\lambda_N^{\alpha_N}$$

where $G(\lambda) = \lambda^\alpha = \lambda_1^{\alpha_1}\cdots\lambda_N^{\alpha_N}$.

**Proposition 3.1.** *Let*

$$\tilde{Z}_d := \int_{F^{m_d^{(N)}}} |VDMH_d((t_1,z^{(1)}),...,(t_{m_d^{(N)}},z^{(m_d^{(N)})}))|^2$$
$$d\nu(t_1,z^{(1)})\cdots d\nu(t_{m_d^{(N)}},z^{(m_d^{(N)})}).$$

*Then* $\tilde{Z}_d = Z_d$ *where* $m_d^{(N)} = \binom{N+d}{d}$ *and (recall (3.6))*

$$Z_d = \int_{E^{m_d^{(N)}}} |VDM(\lambda^{(1)},...,\lambda^{(m_d^{(N)})})|^2 \times$$
$$w(\lambda^{(1)})^{2d}\cdots w(\lambda^{(m_d^{(N)})})^{2d}d\mu(\lambda^{(1)})\cdots d\mu(\lambda^{(m_d^{(N)})}).$$

*Proof.* Recall from section 2 that the $d-$homogeneous Vandermonde determinant $VDMH_d((t_1,z^{(1)}),...,(t_{m_d^{(N)}},z^{(m_d^{(N)})}))$ equals

$$\det\begin{bmatrix} t_1^d & t_2^d & \dots & t_{m_d^{(N)}}^d \\ \vdots & \vdots & \ddots & \vdots \\ e_{m_d^{(N)}}(z^{(1)}) & e_{m_d^{(N)}}(z^{(2)}) & \dots & e_{m_d^{(N)}}(z^{(m_d^{(N)})}) \end{bmatrix}.$$

Expanding this determinant in $\tilde{Z}_d$ gives

$$\tilde{Z}_d = \sum_{I,S}\sigma(I)\cdot\sigma(S)\Big[\int_F t_1^{d-deg(e_{i_1})}e_{i_1}(z^{(1)})\bar{t}_1^{d-deg(e_{s_1})}\overline{e_{s_1}(z^{(1)})}d\nu(t_1,z^{(1)})\cdots$$
$$\cdots\int_F t_{m_d^{(N)}}^{d-deg(e_{i_{m_d^{(N)}}})}e_{i_{m_d^{(N)}}}(z^{(m_d^{(N)})})\bar{t}_{m_d^{(N)}}^{d-deg(e_{s_{m_d^{(N)}}})}\overline{e_{s_{m_d^{(N)}}}(z^{(m_d^{(N)})})}d\nu(t_{m_d^{(N)}},z^{(m_d^{(N)})})\Big]$$

where $I = (i_1, ..., i_{m_d^{(N)}})$ and $S = (s_1, ..., s_{m_d^{(N)}})$ are permutations of $(1, ..., m_d^{(N)})$ and $\sigma(I)$ is the sign of $I$ ($+1$ if $I$ is even; $-1$ if $I$ is odd). Expanding the ordinary Vandermonde determinant in $Z_d$ gives

$$Z_d = \sum_{I,S} \sigma(I) \cdot \sigma(S) \big[ \int_E e_{i_1}(\lambda^{(1)}) \overline{e_{s_1}(\lambda^{(1)})} w(\lambda^{(1)})^{2d} d\mu(\lambda^{(1)}) \cdots$$

$$\cdots \int_E e_{i_{m_d^{(N)}}}(\lambda^{(m_d^{(N)})}) \overline{e_{s_{m_d^{(N)}}}(\lambda^{(m_d^{(N)})})} w(\lambda^{(m_d^{(N)})})^{2d} d\mu(\lambda^{(m_d^{(N)})}) \big].$$

Since $|t_j| = w(\lambda^{(j)})$, using (3.7) completes the proof. □

We need to work in $\mathbb{C}^{N+1}$ with the $\tilde{Z}_d$ integrals and verify the following.

**Proposition 3.2.** *We have*

$$\lim_{n\to\infty} \tilde{Z}_d^{1/2dm_d^{(N)}} = d^{(H)}(F).$$

*Proof.* Fix $d$ and consider the $m_d^{(N)}$ monomials

$$t^d, t^{d-1}z_1, \cdots, z_N^d,$$

utilized in $\tilde{V}DMH_d((t_1, z^{(1)}), ..., (t_n, z^{(m_d^{(N)})}))$. Use Gram-Schmidt in $L^2(\nu)$ to obtain orthogonal homogeneous polynomials

$$q_1^{(H)}(t, z) = t^d,\ q_2^{(H)}(t, z) = t^{d-1}z_1 + ..., \cdots, q_{m_d^{(N)}}^{(H)}(t, z) = z_N^d + ....$$

Then

$$\tilde{V}DMH_d((t_1, z^{(1)}), ..., (t_{m_d^{(N)}}, z^{(m_d^{(N)})})) = \det\big[q_i^{(H)}(t_j, z^{(j)})\big]_{i,j=1,...,m_d^{(N)}}.$$

By orthogonality, as in (3.3), we obtain

$$\tilde{Z}_d = m_d^{(N)}! ||q_1^{(H)}||^2_{L^2(\nu)} \cdots ||q_{m_d^{(N)}}^{(H)}||^2_{L^2(\nu)}.$$

Note that from (2.2) and (2.3) $(m_d^{(N)}!)^{1/2dm_d^{(N)}} \to 1$ as $d \to \infty$. Now from Lemma 2.5 we have

$$\lim_{d\to\infty} \big( \prod_{|\alpha|=d} ||t_{\alpha,F}^{(H)}||_F \big)^{1/dm_d^{(N)}} = \tau(Y_2) = \tau(Y_1) = d^{(H)}(F).$$

Thus we need to show that

$$\lim_{d\to\infty} \big( \prod_{|\alpha|=d} ||t_{\alpha,F}^{(H)}||_F \big)^{1/dm_d^{(N)}} = \lim_{d\to\infty} \big( \prod_{i=1}^{m_d^{(N)}} ||q_i^{(H)}||_{L^2(\nu)} \big)^{1/dm_d^{(N)}}.$$

This is analogous to (3.4) in the proof of Theorem 3.1 and it follows in the same manner from the Bernstein-Markov property for $(F,\nu)$ and the minimality properties of $t_{\alpha,F}^{(H)}$ and $q_i^{(H)}$. $\square$

Combining Propositions 3.1 and 3.2 with equation (2.20) and the second equation in (2.3) completes the proof of Theorem 3.2. $\square$

As a corollary, we get a "large deviation" result, which follows easily from Theorem 3.2. Define a probability measure $\mathcal{P}_d$ on $E^{m_d^{(N)}}$ via, for a Borel set $A \subset E^{m_d^{(N)}}$,

$$\mathcal{P}_d(A) := \frac{1}{Z_d}\int_A |VDM(z_1,...,z_{m_d^{(N)}})|^2 w(z_1)^{2d}\cdots w(z_{m_d^{(N)}})^{2d} d\mu(z_1)\cdots d\mu(z_{m_d^{(N)}}).$$

**Proposition 3.3.** *Given $\eta > 0$, define*

$$A_{d,\eta} := \{(z_1,...,z_{m_d^{(N)}}) \in E^{m_d^{(N)}} :$$

$$|VDM(z_1,...,z_{m_d^{(N)}})|^2 w(z_1)^{2d}\cdots w(z_n)^{2d} \geq (\delta^w(E)-\eta)^{2l_d}\}.$$

*Then there exists $d^* = d^*(\eta)$ such that for all $d > d^*$,*

$$\mathcal{P}_d(E^{m_d^{(N)}} \setminus A_{d,\eta}) \leq (1-\frac{\eta}{2\delta^w(E)})^{2l_d}.$$

*Proof.* From Theorem 3.2, given $\epsilon > 0$,

$$Z_d \geq [\delta^w(E)-\epsilon]^{2l_d}$$

for $d \geq d(\epsilon)$. Thus

$$\mathcal{P}_d(E^{m_d^{(N)}} \setminus A_{d,\eta}) =$$

$$\frac{1}{Z_d}\int_{E^{m_d^{(N)}}\setminus A_{d,\eta}} |VDM(z_1,...,z_{m_d^{(N)}})|^2 w(z_1)^{2d}\cdots w(z_n)^{2d} d\mu(z_1)\cdots d\mu(z_{m_d^{(N)}})$$

$$\leq \frac{[\delta^w(E)-\eta]^{2l_d}}{[\delta^w(E)-\epsilon]^{2l_d}}$$

if $d \geq d(\epsilon)$. Choosing $\epsilon < \eta/2$ and $d^* = d(\epsilon)$ gives the result. $\square$

Finally, we state a a version of (1.2) for $\Gamma$ an unbounded cone in $\mathbb{R}^N$ with $\Gamma = \overline{\text{int}\Gamma}$. Precisely, our set-up is the following. Let $R(x) = R(x_1,...,x_N)$ be a polynomial in $N$ (real) variables $x = (x_1,...,x_N)$ and let

$$d\mu(x) := |R(x)|dx = |R(x_1,...,x_N)|dx_1\cdots dx_N. \tag{3.8}$$

Next, let $w(x) = \exp(-Q(x))$ where $Q(x)$ satisfies the inequality

$$Q(x) \geq c|x|^\gamma \tag{3.9}$$

for all $x \in \Gamma$ for some $c,\gamma > 0$.

**Theorem 3.3.** *Let* $S_w$ :=*supp*$(dd^c V_{\Gamma,Q})^N$ *where* $Q$ *is defined as in (3.9). With* $\mu$ *defined in (3.8),*

$$\lim_{d\to\infty} Z_d^{1/2l_d^{(N)}} = \delta^w(S_w)$$

*where*

$$Z_d = Z_d(\Gamma, w, \mu) := \int_{\Gamma^{m_d^{(N)}}} |VDM(\lambda^{(1)}, ..., \lambda^{(m_d^{(N)})})|^2 \times \tag{3.10}$$

$$w(\lambda^{(1)})^{2d} \cdots w(\lambda^{(m_d^{(N)})})^{2d} d\mu(\lambda^{(1)}) \cdots d\mu(\lambda^{(m_d^{(N)})}).$$

**Remark**. The integrals considered in Theorem 3.3 may be considered as multivariate versions (i.e., with a multivariable Vandermonde determinant in the integrand rather than a one-variable Vandermonde determinant) of integrals of the form

$$\int_{\mathbb{R}^d} |VDM(\lambda_1, ..., \lambda_d)|^2 e^{-dQ(\lambda_1)} \cdots e^{-dQ(\lambda_d)} d\lambda_1 \cdots d\lambda_d$$

considered in [14], Chapter 6, arising in the joint probability distribution of eigenvalues of certain random matrix ensembles. They are also multivariate versions of Selberg integrals of Laguerre type (cf., [18], equation (17.6.5)) which, after rescaling by a factor of $d$, are of the form, for $\Gamma = [0, \infty) \subset \mathbb{R}$ and $\alpha > 0$,

$$\int_{\Gamma^d} |VDM(\lambda_1, ..., \lambda_d)|^2 e^{-d\lambda_1} \cdots e^{-d\lambda_d} (\prod_{j=1}^{d} \lambda_j^{\alpha}) d\lambda_1 \cdots d\lambda_d.$$

*Proof.* We begin by observing that

$$VDM(\lambda^{(1)}, ..., \lambda^{(m_d^{(N)})})^2 \cdot w(\lambda^{(1)})^{2d} \cdots w(\lambda^{(m_d^{(N)})})^{2d} \tag{3.11}$$

(the integrand in (3.10) with the absolute value removed from the VDM) becomes, if all but one of the $m_d^{(N)} - 1$ variables are fixed, a weighted polynomial in the remaining variable. Since $w(x)$ is continuous, by Theorem 2.6 in Appendix B of [20], a weighted polynomial attains its maximum on $S_w \subset \Gamma$. Hence the maximum value of (3.11) on $\Gamma^{m_d^{(N)}}$ is attained on $(S_w)^{m_d^{(N)}}$. Since $S_w$ has compact support (cf., Lemma 2.2 of Appendix B of [20]), we can take $T > 0$ sufficiently large with $S_w \subset \Gamma \cap B_T$ where $B_T := \{x \in \mathbb{R}^N : |x| \leq T\}$ and

$$\delta^w(S_w) = \delta^w(\Gamma \cap B_T).$$

We need the following result.

**Lemma 3.4.** *For all $T>0$ sufficiently large, there exists $M=M(T)>0$ with*
$$||w^d p||_{L^2(\Gamma,\mu)} \leq M||w^d p||_{L^2(\Gamma\cap B_T,\mu)}$$
*if $p=p(x)$ is a polynomial of degree $d$.*

*Proof.* By Theorem 2.6 (ii) in Appendix B of [20], we have
$$|w(x)^d p(x)| \leq ||w^d p||_{S_w} e^{d(V_{\Gamma,Q}(x)-Q(x))}$$
for all $x\in\Gamma$. Since $V_{\Gamma,Q}\in L(\mathbb{C}^N)$ and $Q(x)\geq c|x|^\gamma$ for $x\in\Gamma$, there is a $c_0>0$ with
$$|w(x)^d p(x)| \leq ||w^d p||_{S_w} e^{-c_0 d|x|^\gamma}$$
for all $x\in\Gamma\cap B_T$ for $T$ sufficiently large. Hence
$$||w^d p||_{L^2(\Gamma,\mu)} \leq ||w^d p||_{L^2(\Gamma\cap B_T,\mu)} + ||w^d p||_{S_w}\int_{\Gamma\cap\{|x|\geq T\}} e^{-c_0 d|x|^\gamma}|R(x)|dx.$$
Now $(\Gamma\cap B_T,\mu)$ satisfies the Bernstein-Markov property ([10], Theorem 2.1); thus by [6] Theorem 3.2, the triple $(\Gamma\cap B_T,w,\mu)$ satisfies the weighted Bernstein-Markov property. Thus, given $\epsilon>0$, there is $M_1=M_1(\epsilon)>0$ with
$$||w^d p||_{S_w} = ||w^d p||_{\Gamma\cap B_T} \leq M_1(1+\epsilon)^d||w^d p||_{L^2(\Gamma\cap B_T,\mu)}.$$
A simple estimate shows that
$$\int_{|x|\geq T} e^{-c_0 d|x|^\gamma}|R(x)|dx \leq e^{-c'd}$$
for some $c'>0$. The result now follows by choosing $\epsilon$ sufficiently small. $\square$

We now expand the integrands in the formulas for $Z_d(\Gamma):=Z_d(\Gamma,w,\mu)$ and in $Z_d(\Gamma\cap B_T):=Z_d(\Gamma\cap B_T,w|_{\Gamma\cap B_T},\mu|_{\Gamma\cap B_T})$ as a product of $L^2-$norms of orthogonal polynomials as in (3.3), and then proceed as in the proof of Corollary 2.1 in section 5 of [8] to conclude that
$$\lim_{d\to\infty} Z_d(\Gamma)^{1/2l_d^{(N)}} = \lim_{d\to\infty} Z_d(\Gamma\cap B_T)^{1/2l_d^{(N)}} = \delta^w(\Gamma\cap B_T) = \delta^w(S_w).$$
$\square$

## 4. **Final remarks and questions.**

In this section we discuss some further results in the literature and pose some questions. Recall from section 2 that a $d-$th weighted Fekete set for $E\subset\mathbb{C}^N$ and an admissible weight $w$ on $E$ is a set of $m_d$ points $\zeta_1^{(d)},...,\zeta_{m_d}^{(d)}\in E$ with the property that
$$|W(\zeta_1,...,\zeta_{m_d})| = \sup_{\xi_1,...,\xi_{m_d}\in E}|W(\xi_1,...,\xi_{m_d})|$$

where $W$ is defined in (2.16). In [9] the authors asked if the sequence of probability measures

$$\mu_d := \frac{1}{m_d}\sum_{j=1}^{m_d} < \zeta_j^{(d)} >, \ d=1,2,...,$$

where $< z >$ denotes the point mass at $z$ and $\{\zeta_1^{(d)},...,\zeta_{m_d}^{(d)}\}$ is a $d-$th weighted Fekete set for $E$ and $w$, has a unique weak-* limit, and, if so, whether this limit is the Monge-Ampere measure, $\mu_{eq}^w := (dd^cV_{E,Q}^*)^N$. From the proof of Proposition 2.1, a $d-$th weighted Fekete set for $E$ and $w$ corresponds to a $d-th$ homogeneous Fekete set for the circled set

$$F=F(E,w):=\{(t,z)=(t,t\lambda)\in \mathbb{C}^{N+1}:\lambda\in E, \ |t|=w(\lambda)\};$$

i.e., a set of $m_d^{(N)}=h_d^{(N+1)}$ points in $F$ which maximize the corresponding homogeneous Vandermonde determinant (2.11) for $F$. From Theorem 2.2 of [6], to verify this conjecture for $E\subset \mathbb{C}^N$ and an admissible weight $w$ it suffices to verify it for homogeneous Fekete points associated to circled sets in $\mathbb{C}^{N+1}$.

Suppose now that $\mu$ is a measure on $E$ such that $(E,w,\mu)$ satisfies a Bernstein-Markov inequality for weighted polynomials. Define the probability measures

$$\mu_d(z) := \frac{1}{Z_d}R_1^{(d)}(z)w(z)^{2d}d\mu(z)$$

where $Z_d$ is defined in (3.6) and

$$R_1^{(d)}(z) := \int_{E^{m_d-1}} |VDM(\lambda^{(1)},...,\lambda^{(m_d-1)},z)|^2\cdot \tag{4.1}$$

$$w(\lambda^{(1)})^{2d}\cdots w(\lambda^{(m_d-1)})^{2d}d\mu(\lambda^{(1)})\cdots d\mu(\lambda^{(m_d-1)}).$$

We observe that with the notation in (4.1) and (3.6)

$$\frac{R_1^{(d)}(z)}{Z_d} = \frac{1}{m_d}\sum_{j=1}^{m_d}|q_j^{(d)}(z)|^2 \tag{4.2}$$

where $q_1^{(d)},...,q_{m_d}^{(d)}$ are orthonormal polynomials with respect to the measure $w(z)^{2d}d\mu(z)$ forming a basis for the polynomials of degree at most $d$. To verify (4.2), we refer the reader to the argument in Remark 2.1 of [8]. Forming the sequence of Christoffel functions $K_d(z) := \sum_{j=1}^{m_d}|q_j^{(d)}(z)|^2$, in [8] Theorem 2.2 it was shown that if $N=1$ then $\mu_d(z)\to\mu_{eq}^w(z)$ weak-*; i.e.,

$$\frac{1}{m_d}K_d(z)w(z)^{2d}d\mu(z)\to \mu_{eq}^w(z) \text{ weak-* }. \tag{4.3}$$

We conjecture that (4.3) should hold in $\mathbb{C}^N$ for $N > 1$. To this end, we remark that if $E = \overline{D}$ where $D$ is a smoothly bounded domain in $\mathbb{R}^N$, it follows from the proof of Theorem 1.3 in [1] that $\mu_{eq} := (dd^c V_E^*)^N = c(x)dx$ is absolutely continuous with respect to $\mathbb{R}^N-$Lebesgue measure $dx$ on $D$; and if $\mu(x) = f(x)dx$ is also absolutely continuous, then a conjectured version of (4.3) in the unweighted case $w \equiv 1$ is

$$\frac{1}{m_d} K_d(x) f(x) \to c(x) \text{ on } D.$$

Bos ([12], [13]) has verified this result for centrally symmetric functions $f(x)$ on the unit ball in $\mathbb{R}^N$ and Xu [22] proved this result for certain Jacobi-type functions $f(x)$ on the standard simplex in $\mathbb{R}^N$. For further results on subsets of $\mathbb{R}^1$, see references [10], [13], [15]-[17] in [8]. Berman ([3] and [4]) has shown that if $w = e^{-Q}$ is a smooth admissible weight function on $\mathbb{C}^N$ (recall (2.13)), then $\mu_{eq}^w := (dd^c V_{\mathbb{C}^N,Q}^*)^N = c(z)dz$ is absolutely continuous with respect to $\mathbb{C}^N-$Lebesgue measure $dz$ on the interior $I$ of the compact set $\{z \in \mathbb{C}^N : V_{\mathbb{C}^N,Q}(z) = Q(z)\}$ and

$$\frac{1}{m_d} K_d(z) Q(z)^d \to c(z) \text{ a.e. on } I.$$

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